The Invariant Ring of Triples of 3×3 Matrices over a Field of Arbitrary Characteristic

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Abstract

Let $R_{n,d}$ be the ring of invariants of d-tuples of $n \times n$ matrices under the simultaneous conjugation action of the general linear group. A minimal generating system and a homogeneous system of parameters of $R_{3,3}$ are determined. Homogeneous systems of parameters of $R_{3,2}$, $R_{4,2}$ are also pointed out.

1 Introduction

Let K be an infinite field of the characteristic p ($p = 0, 2, 3, \ldots$). Denote by $M_{n,d}(K) = M_n(K) \oplus \cdots \oplus M_n(K)$ the space of d-tuples of $n \times n$ matrices, where $d \geq 1$. The general linear group $GL_n(K)$ acts by simultaneous conjugation on $M_{n,d}(K)$: for $g \in GL_n(K)$, $A_i \in M_n(K)$ ($i = \overline{1,d}$), we have

$$g(A_1, \ldots, A_d) = (gA_1g^{-1}, \ldots, gA_dg^{-1}).$$

The coordinate ring of $M_{n,d}(K)$ is the polynomial algebra $K_{n,d} = K[x_{ij}(r)\,|1 \leq i,j \leq n,\ r = \overline{1,d}]$, where $x_{ij}(r)$ denotes the function mapping $(A_1, \ldots, A_d) \in M_{n,d}(K)$ to the (i,j) entry of A_r. The action of $GL_n(K)$ on $M_{n,d}(K)$ induces the action of $GL_n(K)$ on $K_{n,d}$: $(g \cdot f)(A) = f(g^{-1}A)$, where $g \in GL_n(K)$, $f \in K_{n,d}$, $A \in M_{n,d}$. Denote by

$$R_{n,d} = \{f \in K_{n,d}\,|\forall g \in GL_n(K) : gf = f\}$$

the matrix algebra of invariants. Let $X_r = (x_{ij}(r))_{1\leq i,j\leq n}$ be the generic matrices of order n ($r=\overline{1,d}$), and let $\sigma_k(A)$ be the coefficients of the characteristic polynomial of $A\in M_n(K)$, i.e.,

$$\det(\lambda E - A) = \lambda^n - \sigma_1(A)\lambda^{n-1} + \cdots + (-1)^n\sigma_n(A).$$

Here E stands for the identity matrix. The algebra $R_{n,d}$ is generated by all elements of the form $\sigma_k(X_{i_1}\cdots X_{i_s})$ [1]. The Procesi–Razmyslov Theorem on relations of $R_{n,d}$ was extended to the case of arbitrary p in [2].

Let $N_0 = \{0,1,2,\ldots\}$. The algebra $R_{n,d}$ possesses the natural N_0-grading by degrees and N_0^d-grading by multidegrees. If some N_0^d-homogeneous subset $G\subset R_{n,d}$ has the property that G generates $R_{n,d}$ but any proper subset of G does not generate $R_{n,d}$, then G is called a homogeneous minimal system of generators (shortly h.m.s.o.g.) of $R_{n,d}$. A h.m.s.o.g. of $R_{2,d}$ was found in [3] when $p=0$, in [4] when $p>2$, and in [5] when $p=2$. In [6] some upper and lower bounds on the highest degree of elements of a h.m.s.o.g. of $R_{n,d}$ are pointed out for arbitrary p. In [7] in case of $p=0$ the cardinality of a h.m.s.o.g. of $R_{3,d}$ was calculated for $d\leq 10$ by means a computer, and shown a way how the set can be constructed for each d by means a computer. The least upper bound on degrees of elements of a h.m.s.o.g. of $R_{3,d}$ was established in [8] (except for the case of $p=3$, $d=6k+1$, $k>0$, where error of estimation of the least upper bound is not greater than 1). In the given paper we construct a h.m.s.o.g. of $R_{3,3}$ for arbitrary p.

By the Noether normalization $R_{n,d}$ contains a *homogeneous* (with respect to N_0-grading) *system of parameters* (shortly h.s.o.p), i.e. a set of algebraically independent elements generating a subalgebra over which $R_{n,d}$ is integral. In case of $p=0$ Teranishi established h.s.o.p.-s of $R_{3,2}$, $R_{4,2}$ [9] and of $R_{2,d}$ ($d\geq 2$). In [5] h.s.o.p.-s were established for $R_{2,d}$ for $p>0$, $d\geq 2$. The main result of the paper is the explicit description of a h.s.o.p. for $R_{3,3}$ over a field of arbitrary characteristic. We also point out h.s.o.p.-s for $R_{3,2}$, $R_{4,2}$ for arbitrary p.

2 Auxiliary results

Let S be the free semigroup generated by letters $\{x_1,x_2,\ldots\}$. For a word $u\in S$ we denote the degree of u by $\deg(u)$, the multidegree of u by $\mathrm{mdeg}(u)$, and the degree of u with respect to the letter x_j by $\deg_{x_j}(u)$. All word are supposed to be non-empty unless otherwise is stated.

Denote by $R_{n,d}^{+}$ the subalgebra generated by elements of $R_{n,d}$ of positive degree. An element $r\in R_{n,d}$ is called *decomposable* if it can be get in terms of elements

whose degrees are less than that of r, i.e. r belongs to the ideal $(R^+_{n,d})^2$. Obviously, $\{r_i\} \in R_{n,d}$ is h.m.s.o.g. iff $\{\overline{r_i}\}$ is a base of $\overline{R_{n,d}} = R_{n,d}/(R^+_{n,d})^2$. If elements r_1, r_2 of $R_{n,d}$ are equal modulo $(R^+_{n,d})^2$, we write $r_1 \equiv r_2$. Denote by $K\langle x_1, \ldots, x_d\rangle^\#$ the free associative K-algebra without unity which is freely generated by $x_1, \ldots, x_d$. Let $\mathrm{id}\{f_1, \ldots, f_s\}$ be the ideal generated by $f_1, \ldots, f_s$. The problem of decomposability of an element of $R_{n,d}$ and the problem of equality to zero of some element of $N_{n,d} = K\langle x_1, \ldots, x_d\rangle^\# / \mathrm{id}\{x^n \mid x \in K\langle x_1, \ldots, x_d\rangle^\#\}$ are closely related (see Lemma 2 below). Let $A_{n,d}$ be K-algebra without unity generated by generic matrices $X_1, \ldots, X_d$. The homomorphism of algebras $\phi : A_{n,d} \to N_{n,d}$, which maps X_i in x_i, is defined correctly.

A word $w \in S$ is called *canonical* with respect to x_i, if it has one of the following forms: w_1, $w_1 x_i w_2$, $w_1 x_i^2 w_2$, $w_1 x_i^2 u x_i w_2$, where words w_1, w_2, u do not contain x_i, words w_1, w_2 can be empty. If a word is canonical with respect to all letters, then we call it *canonical*. In [8] the next lemmas are proved.

Lemma 1 *1. Applying identities $x_i u x_i = -x_i^2 u - u x_i^2$, $x_i u x_i^2 = -x_i^2 u x_i$ of $N_{3,d}$, every non-zero word $w \in N_{3,d}$ can be represented as a sum of canonical words which belong to the same homogeneous component as w.*

2. If $p \neq 3$, then $x_i^2 u x_j^2 = 0$ holds in $N_{3,d}$.

3. The equality $x_j^2 x_i^2 x_j x_i = -x_i^2 x_j^2 x_i x_j$ holds in $N_{3,d}$.

4. The inequality $x_1^2 x_2^2 x_1 \neq 0$ holds in $N_{3,d}$.

5. Let $p = 3$ and for some identity $\sum \alpha_i u_i = 0$ of $N_{3,d}$, $\alpha_i \in K$, exists k such that $\deg_{x_k}(u_i) = 1, 2$. Then after the substitution $x_k = 1$ in $\sum \alpha_i u_i = 0$ we get an identity of $N_{3,d}$.

Lemma 2 *1. Let $G \in A_{n,d}$, $i = \overline{1,d}$. Then*

a) if G do not contain X_i and $\mathrm{tr}(GX_i)$ is decomposable, then $\phi(G) = 0$;

b) if $\phi(G) = 0$, then $\mathrm{tr}(GX_i)$ is decomposable.

2. Let $G \in A_{n,d}$ and G do not contain X_i for some $i = \overline{1,d}$. Then decomposability of $\mathrm{tr}(GX_i^2)$ implies $\phi(G)x_i + x_i\phi(G) = 0$ in $N_{3,d}$.

3. Let $U \in A_{n,d}$ be a word. If $\mathrm{tr}(U)$ is indecomposable, then $\mathrm{tr}(U)$ can be represented in the form $\mathrm{tr}(U) \equiv \sum \alpha_i \mathrm{tr}(W_i)$, where words $W_i \in A_{n,d}$ are canonical, $\mathrm{mdeg}(U) = \mathrm{mdeg}(W_i)$, $\alpha_i \in K$.

4. The identity $\sigma_2(UV) \equiv \mathrm{tr}(U^2V^2)$, where $U, V \in A_{n,d}$, holds in $R_{3,d}$.

5. The explicit upper bound on degrees of elements of a h.m.s.o.g. of $R_{3,3}$ is equal to 6, if $p \neq 3$, and is equal to 8, if $p = 3$.

6. The explicit upper bound on degrees of elements of a h.m.s.o.g. of $R_{3,d}$ ($d \geq 2$) is equal to 6, if $p = 0$.

The result of substitution $X_1 \to A_1, \dots, X_d \to A_d$ in $r \in R_{n,d}$, where $A_1, \dots, A_d \in M_n(K)$ ($A_1, \dots, A_d$ are some generic matrices, respectively), denote by $r|_{A_1,\dots,A_d}$. If we consider a subset of $R_{n,d}$ instead of r, then we use the same notation. If $Q \subset R_{n,d}$, $A_i \in M_n(K)$, $i = \overline{1,d}$, and all elements of $Q|_{A_1,\dots,A_d}$ are equal to zero, then we write $Q|_{A_1,\dots,A_d} = 0$.

If $A, B \in M_n(K)$ are equivalent, i.e. there is $T \in GL_n(K)$ such that $A = TBT^{-1}$, then we write $A \sim B$. Let $\mathrm{rank}(A)$ be the rank of a matrix A. Denote

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Further n is assumed to be 3 unless otherwise is stated.

3 A homogeneous minimal system of generators

Let $i, j, k = \overline{1,3}$ be pairwise different. Denote by G_1 the set

$$\begin{array}{l} \mathrm{tr}(X_i); \\ \mathrm{tr}(X_iX_j),\ i<j;\ \sigma_2(X_i); \\ \mathrm{tr}(X_1X_2X_3),\ \mathrm{tr}(X_1X_3X_2);\ \mathrm{tr}(X_i^2X_j);\ \sigma_3(X_i); \\ \mathrm{tr}(X_i^2X_j^2),\ i<j;\ \mathrm{tr}(X_i^2X_jX_k); \\ \mathrm{tr}(X_i^2X_j^2X_k);\ \mathrm{tr}(X_i^2X_jX_iX_k),\ j<k. \end{array}$$

Denote by G_2 the set

$$\mathrm{tr}(X_i^2X_j^2X_iX_j),\ i<j;\ \mathrm{tr}(X_i^2X_j^2X_iX_k);\ \mathrm{tr}(X_1^2X_2^2X_3^2).$$

Denote by G_3 the set

$$\begin{array}{l} \mathrm{tr}(X_i^2X_j^2X_iX_j),\ i<j;\ \mathrm{tr}(X_i^2X_j^2X_iX_k);\ \mathrm{tr}(X_1^2X_2^2X_3^2),\ \mathrm{tr}(X_1^2X_3^2X_2^2); \\ \mathrm{tr}(X_iX_j^2X_k^2X_jX_k),\ j<k;\ \mathrm{tr}(X_i^2X_j^2X_iX_k^2),\ j<k; \\ \mathrm{tr}(X_i^2X_j^2X_k^2X_jX_k),\ j<k. \end{array}$$

Theorem 1 *(i) If $p \neq 3$, then $G_{\mathrm{i}} = G_1 \cup G_2$ is a homogeneous minimal system of generators of $R_{3,3}$.*
(ii) If $p = 3$, then $G_{\mathrm{ii}} = G_1 \cup G_3$ is a homogeneous minimal system of generators of $R_{3,3}$.

Remark 1 *We have $|G_{\mathrm{i}}| = 48$, $|G_{\mathrm{ii}}| = 58$.*

In order to prove the Theorem we formulate the next Lemma.

Lemma 3 *1. The elements* $\mathrm{tr}(X_1X_2X_3)$, $\mathrm{tr}(X_1X_3X_2)$ *are linearly independent in* $\overline{R_{3,3}}$.
2. The elements $\mathrm{tr}(X_1^2X_2X_3)$, $\mathrm{tr}(X_1^2X_3X_2)$ *are linearly independent in* $\overline{R_{3,3}}$.
3. The elements $\mathrm{tr}(X_1^2X_2^2X_3)$, $\mathrm{tr}(X_2^2X_1^2X_3)$ *are linearly independent in* $\overline{R_{3,3}}$.
4. If $p=3$*, then the elements* $\mathrm{tr}(X_1^2X_2^2X_3^2)$, $\mathrm{tr}(X_1^2X_3^2X_2^2)$ *are linearly independent in* $\overline{R_{3,3}}$. *If* $p\neq 3$*, then they are linearly dependent in* $\overline{R_{3,3}}$.

Proof. Items 1, 2 are consequences of item 3, because if $\sum_i \mathrm{tr}(U_iX_j)\equiv 0$, where words $U_i\in A_{n,d}$ and $\deg_{X_j}(U_i)=0$, then $\sum_i \mathrm{tr}(U_iX_j^2)\equiv 0$.

3. Let $\alpha\mathrm{tr}(X_1^2X_2^2X_3)+\beta\mathrm{tr}(X_2^2X_1^2X_3)\equiv 0$, where $\alpha\neq 0$ or $\beta\neq 0$. Hence $\alpha x_1^2x_2^2+\beta x_2^2x_1^2=0$ in $N_{3,d}$ by item 1 of Lemma 2. Since $x_2^2x_1^2\neq 0$ in $N_{3,d}$ (item 4 of Lemma 1), we have $\alpha\neq 0$. Thus $x_1^2x_2^2x_1=-(\beta/\alpha)x_2^2x_1^3=0$ in $N_{3,d}$, but $x_1^2x_2^2x_1\neq 0$ in $N_{3,d}$ (item 4 of Lemma 1); a contradiction.

4. Let $p=3$ and $\alpha\mathrm{tr}(X_1^2X_2^2X_3^2)+\beta\mathrm{tr}(X_2^2X_1^2X_3^2)\equiv 0$, where $\alpha\neq 0$ or $\beta\neq 0$. Hence $(\alpha x_1^2x_2^2+\beta x_2^2x_1^2)x_3+x_3(\alpha x_1^2x_2^2+\beta x_2^2x_1^2)=0$ in $N_{3,d}$ (item 2 of Lemma 2). Substitution $x_3=1$ gives $2\alpha x_1^2x_2^2+2\beta x_2^2x_1^2=0$ (item 5 of Lemma 1), which was proved to be a contradiction.

Let $p\neq 3$. The identity $x_1^2x_2x_3^2=0$ in $N_{3,d}$ (item 2 of Lemma 1) implies $\mathrm{tr}(X_1^2X_2X_3^2X_2)\equiv 0$ (item 1 of Lemma 2). On the other hand, $x_2x_3^2x_2=-x_2^2x_3^2-x_3^2x_2^2$ in $N_{3,d}$ (item 1 of Lemma 1) implies $\mathrm{tr}(X_1^2X_2X_3^2X_2)\equiv -\mathrm{tr}(X_1^2X_2^2X_3^2)-\mathrm{tr}(X_1^2X_3^2X_2^2)$ (item 1 of Lemma 2). $\triangle$

Proof of Theorem 1. Lemmas 1, 2 imply that G_{i} (G_{ii}, respectively) generates $R_{3,3}$ when $p\neq 3$ ($p=3$, respectively). These lemmas also show that all elements of G_{i} (of G_{ii}, respectively) are indecomposable, when $p\neq 3$ ($p=3$, respectively). Thus it is enough to prove that the elements of G_{i} (of G_{ii}, respectively) of the equal multidegree are linearly independent in $\overline{R_{3,3}}$. The last follows from Lemma 3. $\triangle$

The next proposition follows easily from [7].

Proposition 1 *Let* $p=0$. *Then the cardinality of a h.m.s.o.g. of* $R_{3,d}$ *equals to* $M_d=3d+5\binom{d}{2}+24\binom{d}{3}+51\binom{d}{4}+47\binom{d}{5}+15\binom{d}{6}$, *where* $\binom{d}{i}=0$ *for* $i>d$.

Proof. Consider some h.m.s.o.g. of $R_{3,d}$. Denote by G_d its subset which consists of elements depending on $X_1,\ldots,X_d$. It is easy to see, that

$$\bigcup_{k=1}^{d}\ \bigcup_{1\leq i_1<\cdots<i_k\leq d} G_k|_{X_{i_1},\ldots,X_{i_k}}$$

is a h.m.s.o.g. of $R_{3,d}$. Thus $M_d = \sum_{k=1}^{d} a_k \binom{d}{k}$, where $a_k = |G_k|$. The algebra $R_{3,d}$ is generated by elements of the degree not greater than 6 (for example, see item 6 of Lemma 2). Hence $a_i = 0$ for $i > 6$. In [7] it was shown that

d	1	2	3	4	5	6
M_d	3	11	48	189	607	1635

.

Thus we get the required. △

4 A homogeneous system of parameters

Theorem 2 *The set $P \subset R_{3,3}$*

$$\begin{array}{l}
\sigma_k(X_i),\ i,k=\overline{1,3},\\
\mathrm{tr}(X_1X_2),\ \mathrm{tr}(X_1X_3),\ \mathrm{tr}(X_2X_3),\\
\mathrm{tr}(X_1^2X_2)+\alpha_1\mathrm{tr}(X_2^2X_3)+\alpha_2\mathrm{tr}(X_3^2X_1),\\
\mathrm{tr}(X_1^2X_3)-\beta_1\mathrm{tr}(X_3^2X_2),\ \mathrm{tr}(X_1^2X_3)-\beta_2\mathrm{tr}(X_2^2X_1),\\
\mathrm{tr}(X_1X_2X_3)+\gamma\mathrm{tr}(X_1X_3X_2),\\
\mathrm{tr}(X_1^2X_2^2),\ \mathrm{tr}(X_1^2X_3^2),\ \mathrm{tr}(X_2^2X_3^2),
\end{array}$$

where α_1, α_2, β_1, β_2, γ are non-zero elements of K and $\alpha_1+\beta_1+\alpha_2\beta_2 \neq 0$, is a homogeneous system of parameters of $R_{3,3}$.

It is easy to see that if the statement of the Theorem is valid over the algebraic closure of K, then it is valid over K. Therefore we can assume that K is algebraically closed.

We need the following Theorem, which was proved by Hilbert for $p=0$ [11]. In [12] there is a proof of the Theorem for $p=0$, and that proof is suited for arbitrary p.

Let an algebraic group G acts regularly on some affine variety X. This action induces the action of G on the coordinate ring $K[X]$ which consists of regular mappings from X into K. Let invariants $I_1,\ldots,I_s \in K[X]^G$ have the property: if $I_1(x)=\cdots=I_s(x)=0$, where $x\in X$, then for each homogeneous non-constant invariant $I\in K[X]^G$ we have $I(x)=0$. Then the ring of invariants $K[X]^G$ is integral over the subring generated by $I_1,\ldots,I_s$.

Denote by $\mathrm{tr.deg}(R_{n,d})$ the transcendence degree of $R_{n,d}$ ($n\geq 2$), i.e. the cardinality of its h.m.s.o.g. The next Lemma is a folklore statement but we give its proof for completeness.

Lemma 4 $\text{tr.deg}(R_{n,d}) = (d-1)n^2 + 1$, *where* $d \geq 2$.

Proof. Recall the Theorem whose proof can be found, for example, in [13].

If $\phi : X \to Y$ *is a dominant morphism of irreducible algebraic varieties, then there is an open non-empty subspace* $U \subset Y$ *such that for all* $y \in U$ *we have* $\dim(\phi^{-1}(y)) = \dim(X) - \dim(Y)$.

Let $\pi : M_{n,d}(K) \to M_{n,d}(K)/GL_n(K) = \text{Spec}(R_{n,d})$ be categorical quotient, where $\text{Spec}(R_{n,d})$ is the affine variety whose coordinate ring is isomorphic to $R_{n,d}$. Denote by F_d the free associative algebra generated freely by $f_1, \ldots, f_d$. Each representation $\Psi : F_d \to M_n(K)$ correspond to some point $(\Psi(f_1), \ldots, \Psi(f_d)) \in M_{n,d}(K)$ and vice visa. Denote by W the set of points of $M_{n,d}(K)$ which correspond to the simple representations. It is known that if a point belongs to W, then its $GL_n(K)$-orbit is closed and coincides with the fiber of π which contains it (see [14], [15]).

By Frobenius theorem representation Ψ is simple iff there are $g_1, \ldots, g_{n^2} \in F_d$ such that $\Psi(g_1), \ldots, \Psi(g_{n^2})$ are linearly independent. Thus Ψ is simple iff

$$\delta = \det \begin{pmatrix} \Psi(g_1)_1 & \cdots & \Psi(g_1)_{n^2} \\ \vdots & & \vdots \\ \Psi(g_{n^2})_1 & \cdots & \Psi(g_{n^2})_{n^2} \end{pmatrix} \neq 0,$$

where $\Psi(g_i)_j$ stands for the j^{th} entry of matrix $\Psi(g_i)$. The number δ is equal to the value of some polynomial $h_{g_1,\ldots,g_{n^2}} \in K_{n,d}$, depending on $g_1, \ldots, g_{n^2}$, calculated at the point $x_{ij}(r) = \Psi(f_r)_{ij}$, where $\Psi(f_r)_{ij}$ is the (i,j) entry of the matrix $\Psi(f_r)$, $i,j = \overline{1,n}$, $r = \overline{1,d}$. For $h \in K_{n,d}$ denote $\overline{V}(h) = \{A \in M_{n,d}(K) | h(A) \neq 0\}$. We have

$$W = \bigcup_{g_1,\ldots,g_{n^2} \in F_d} \overline{V}(h_{g_1,\ldots,g_{n^2}})$$

is an open set.

Apply the Theorem mentioned to π. Since $M_{n,d}(K)$ is irreducible, there is $x \in \pi^{-1}(U) \cap W$. Thus there exists $y \in U$ such that $x \in \pi^{-1}(y)$. Hence $\mathrm{O}(x) = \pi^{-1}(y)$, where $\mathrm{O}(x)$ denotes the orbit of x. We have $\dim \mathrm{O}(x) = \dim GL_n(K) - \dim \text{St}(x)$, where $\text{St}(x)$ stands for the stabilizer of x. Since $\dim GL_n(K) = n^2$, $\dim \text{St}(x) = 1$, $\dim M_{n,d} = n^2 d$ and $\dim \text{Spec}(R_{n,d}) = \text{tr.deg}(R_{n,d})$, the Lemma is proved. $\triangle$

Lemma 5 *Let* $A \in M_3(K)$ *and* $\sigma_k(A) = 0$ *(*$k = \overline{1,3}$*). Then* $\text{rank}(A) \leq 2$. *Moreover,*

$$\begin{array}{lll} \text{rank}(A) = 0 & \text{iff} & A = 0, \\ \text{rank}(A) = 1 & \text{iff} & A \sim J_1, \\ \text{rank}(A) = 2 & \text{iff} & A \sim J_2, \\ \text{rank}(A) \leq 1 & \text{iff} & A^2 = 0. \end{array}$$

Proof. Lemma follows from the Theorem on Jordan form of matrix and the Cayley–Hamilton identity: $A^3-\mathrm{tr}(A)A^2+\sigma_2(A)A-\det(A)E=0$, where $A\in M_3(K)$. $\triangle$

Lemma 6 *Let* $A=J_2$, $B\in M_3(K)$, $\mathrm{rank}(B)=2$, $\sigma_k(B)=0$ *(*$k=\overline{1,3}$*)*, $\mathrm{tr}(AB)=0$, $\mathrm{tr}(A^2B^2)=0$. *Then one of the following possibilities is valid:*

(1) $B=\begin{pmatrix} 0 & t_1 & t_2\\ 0 & 0 & t_3\\ 0 & 0 & 0\end{pmatrix}$.

(2) $B=TAT^{-1}$, $T=\begin{pmatrix} t_3t_4 & t_1 & t_2\\ t_3 & t_4 & t_5\\ 0 & 1 & 0\end{pmatrix}$, *where* $\det(T)\neq 0$.

(3) $B=TAT^{-1}$, $T=\begin{pmatrix} t_1 & t_2 & t_3t_4\\ t_3 & 0 & t_4\\ 1 & 0 & 0\end{pmatrix}$, *where* $\det(T)\neq 0$.

Proof. Lemma 5 implies $B=TJ_2T^{-1}$ for some $T\in GL_3(K)$. Let

$$L=\begin{pmatrix} a & b & c\\ 0 & a & b\\ 0 & 0 & a\end{pmatrix},$$

where $a,b,c\in K$, $a\neq 0$. We have $TJ_2T^{-1}=(TL)J_2(TL)^{-1}$. There are a,b,c ($a\neq 0$) such that TL is equal to one of the following matrices:

$$\begin{pmatrix} * & * & *\\ * & * & *\\ 0 & 0 & 1\end{pmatrix},\quad \begin{pmatrix} * & * & *\\ * & * & *\\ 0 & 1 & 0\end{pmatrix},\quad \begin{pmatrix} * & * & *\\ * & * & *\\ 1 & 0 & 0\end{pmatrix}.$$

Substituting TL for T and considering the equations $\mathrm{tr}(AB)=\mathrm{tr}(A^2B^2)=0$ we get the required. $\triangle$

Lemma 7 *Let* $A=J_1$, $\sigma_k(B)=0$ *(*$k=\overline{1,3}$*)*, $\mathrm{tr}(AB)=0$, $\mathrm{rank}(B)=1$. *Then* $AB=0$ *or* $BA=0$.

Proof. Let $B=(b_{ij})_{1\le i,j\le 3}$. The equation $\mathrm{tr}(AB)=0$ implies $b_{21}=0$. Lemma 5 states $B^2=0$. Thus we obtain $b_{23}b_{31}=0$. Hence we have three alternatives: $b_{23}=0$, $b_{31}\neq 0$, or $b_{23}\neq 0$, $b_{31}=0$, or $b_{23}=b_{31}=0$. Performing direct computations and making appropriate substitutions we get that one of the following possibilities is valid:

(1) $B = \begin{pmatrix} b_3 & b_2b_3/b_1 & -b_3^2/b_1 \\ 0 & 0 & 0 \\ b_1 & b_2 & -b_3 \end{pmatrix}$, where $b_1 \neq 0$.

(2) $B = \begin{pmatrix} 0 & b_1b_3/b_2 & b_1 \\ 0 & b_3 & b_2 \\ 0 & -b_3^2/b_2 & -b_3 \end{pmatrix}$, where $b_2 \neq 0$.

(3) $B = \begin{pmatrix} 0 & b_1 & b_2 \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix}$, where $b_2 = 0$ or $b_3 = 0$.

The Lemma is proved. $\triangle$

The next Lemma was proved in [9].

Lemma 8 *Let $A, B \in M_3(K)$, $\mathrm{rank}(A) = 2$, $\sigma_k(A) = \sigma_k(B) = 0$ ($k = \overline{1,3}$), and $\mathrm{tr}(AB) = \mathrm{tr}(A^2B^2) = \mathrm{tr}(A^2B) = 0$. Then $\mathrm{tr}(AB^2) = 0$. If also $A = J_2$, then B is a strictly upper triangular matrix.*

Proof. Conjugating, we can suppose that $A = J_2$ and $B = (b_{ij})_{1\leq i,j\leq 3}$. The equation $\mathrm{tr}(A^2B) = 0$ implies $b_{31} = 0$, and $\mathrm{tr}(AB) = b_{21} + b_{32} = 0$, $\mathrm{tr}(A^2B^2) = b_{21}b_{32} = 0$ imply $b_{21} = b_{32} = 0$. It follows from $\sigma_k(B) = 0$, $k = \overline{1,3}$, that $b_{11} = b_{22} = b_{33} = 0$. Thus B is a strictly upper triangular matrix and $\mathrm{tr}(AB^2) = 0$. $\triangle$

Lemma 9 *Let $Q \subset R_{3,d}$ be the set*

$$\begin{array}{ll} \sigma_k(X_i),\ k = \overline{1,3},\ i = \overline{1,d}, & \mathrm{tr}(X_i^2X_j),\ 1 \leq i \neq j \leq d, \\ \mathrm{tr}(X_i^2X_j^2),\ 1 \leq i < j \leq d, & \mathrm{tr}(X_{\sigma(1)}\cdots X_{\sigma(r)}),\ r = \overline{2,d},\ \sigma \in S_r. \end{array}$$

If $A_1, \ldots, A_s \in M_3(K)$ are such that $Q|_{A_1,\ldots,A_d} = 0$, then for each $r \in R_{3,d}^+$ we have $r|_{A_1,\ldots,A_d} = 0$.

Proof. If there is $i = \overline{1,d}$ such that $A_i = 0$, then the statement is proved by induction on d.

Let $\mathrm{rank}(A_j) = 2$ for some $j = \overline{1,d}$. Conjugating, we can suppose $A_j = J_2$. Lemma 8 implies that every A_i is a strictly upper triangular matrix. Thus the required is proved.

The only case we has not yet considered is $\mathrm{rank}(A_i) = 1$, $i = \overline{1,d}$ (see Lemma 5). Direct computations yields $J_1BJ_1 = \mathrm{tr}(J_1B)J_1$, $B \in M_3(K)$, and conjugating the equality we get $A_iBA_i = \mathrm{tr}(A_iB)A_i$, $B \in M_3(K)$, $i = \overline{1,d}$. The latter identity

implies that if V is a word in matrices $A_1,\ldots,A_d$ and $\deg_{A_j}(V)\geq 2$ for some j, then $V=0$. Thus we has shown that $\mathrm{tr}(U)=0$ for each word U in $A_1,\ldots,A_d$.

Prove by induction on $\deg(V)$ that $\sigma_2(V)=0$, where V is a word in matrices $A_1,\ldots,A_d$. We have $\sigma_2(UV)\equiv \mathrm{tr}(U^2V^2)$ (item 4 of Lemma 2), where U,V are words in $A_1,\ldots,A_d$. By induction hypothesis we have $\sigma_2(U)=\sigma_2(V)=0$, which together with proved part of the Lemma gives $\sigma_2(UV)=0$.

So the Lemma is proved. $\triangle$

Lemma 10 *Let $Q\subset R_{3,3}$ be the set*

$$\begin{array}{l}\sigma_k(X_i),\ k,i=\overline{1,3},\quad \mathrm{tr}(X_iX_j),\ 1\leq i<j\leq 3,\quad \mathrm{tr}(X_i^2X_j),\ 1\leq i\neq j\leq 3,\\ \mathrm{tr}(X_i^2X_j^2),\ 1\leq i<j\leq 3,\quad \alpha\mathrm{tr}(X_1X_2X_3)+\beta\mathrm{tr}(X_1X_3X_2),\end{array}$$

where $\alpha,\beta\in K$ are non-zero. If $A_1,A_2,A_3\in M_3(K)$ are such that $Q|_{A_1,A_2,A_3}=0$, then for each $r\in R_{3,3}^{+}$ we have $r|_{A_1,A_2,A_3}=0$.

Proof. If there is $j=\overline{1,3}$ such that $A_j=0$, then Lemma 9 concludes the proof.

If there is $j=\overline{1,3}$ such that $\mathrm{rank}(A_j)=2$, then conjugating A_1,A_2,A_3 we can assume that $A_j=J_2$ (Lemma 5). Lemma 8 implies that A_1,A_2,A_3 are strictly upper triangular matrices. Thus the required is proved.

Let $\mathrm{rank}(A_i)=1$, $i=\overline{1,3}$. Conjugating, we can assume $A=J_1$. Lemma 7 implies $A_1A_2=0$ or $A_2A_1=0$. Thus $\mathrm{tr}(A_1A_2A_3)=0$ or $\mathrm{tr}(A_1A_3A_2)=0$. Lemma 9 concludes the prove.

By Lemma 5, all possibilities have been considered. $\triangle$

Lemma 11 *Let $Q\subset R_{3,3}$ be the set*

$$\begin{array}{l}\sigma_k(X_i),\ k,i=\overline{1,3},\quad \mathrm{tr}(X_iX_j),\ 1\leq i<j\leq 3,\quad \mathrm{tr}(X_i^2X_j^2),\ 1\leq i<j\leq 3,\\ \mathrm{tr}(X_1^2X_3),\quad \mathrm{tr}(X_3^2X_2),\quad \mathrm{tr}(X_2^2X_1),\\ \alpha_1\mathrm{tr}(X_1^2X_2)+\alpha_2\mathrm{tr}(X_2^2X_3)+\alpha_3\mathrm{tr}(X_3^2X_1),\\ \beta_1\mathrm{tr}(X_1X_2X_3)+\beta_2\mathrm{tr}(X_1X_3X_2),\end{array}$$

where $\alpha_1,\ldots,\beta_2\in K$ are non-zero. If $A_1,A_2,A_3\in M_3(K)$ are such that $Q|_{A_1,A_2,A_3}=0$, then for each $r\in R_{3,3}^{+}$ we have $r|_{A_1,A_2,A_3}=0$.

Proof. If there is $j=\overline{1,3}$ such that $A_j=0$, then Lemma 9 concludes the proof.

Let $\mathrm{rank}(A_1)=\mathrm{rank}(A_2)=2$. Applying Lemma 8 to matrices A_1, A_3 and A_2, A_1 we obtain that $\mathrm{tr}(A_3^2A_1)=0$ and $\mathrm{tr}(A_1^2A_2)=0$. Thus $\alpha_2\mathrm{tr}(A_2^2A_3)=0$. The statement follows from Lemma 10.

Let $\mathrm{rank}(A_1)=\mathrm{rank}(A_2)=1$. Thus $A_1^2=A_2^2=0$ (Lemma 5). Hence $\mathrm{tr}(A_1^2A_2)=\mathrm{tr}(A_2^2A_3)=0$, and thus $\alpha_3\mathrm{tr}(A_3^2A_1)=0$. The statement follows from Lemma 10.

By symmetry and Lemma 5, all possibilities have been considered. $\triangle$

Proof of Theorem 2. In order to prove the Theorem it is sufficient to show that if $A_1,A_2,A_3\in M_3(K)$ are such that $P|_{A_1,A_2,A_3}=0$, then for each $r\in R^+_{3,3}$ we have $r|_{A_1,A_2,A_3}=0$ (see Lemma 4 and the above mentioned Hilbert Theorem). For convenience enumerate the equations:

$$\mathrm{tr}(A_2A_3)=0. \tag{1}$$

$$\mathrm{tr}(A_1^2A_2)+\alpha_1\mathrm{tr}(A_2^2A_3)+\alpha_2\mathrm{tr}(A_3^2A_1)=0. \tag{2}$$

$$\mathrm{tr}(A_1^2A_3)-\beta_1\mathrm{tr}(A_3^2A_2)=0. \tag{3}$$

$$\mathrm{tr}(A_1^2A_3)-\beta_2\mathrm{tr}(A_2^2A_1)=0. \tag{4}$$

$$\mathrm{tr}(A_1A_2A_3)+\gamma\mathrm{tr}(A_1A_3A_2)=0. \tag{5}$$

$$\mathrm{tr}(A_2^2A_3^2)=0. \tag{6}$$

If there is $j=\overline{1,3}$ such that $A_j=0$, then Lemma 9 concludes the proof.

Let $\mathrm{rank}(A_j)=1$ for some $j=\overline{1,3}$. Thus $A_j^2=0$ (Lemma 5). The equations (3), (4) imply that $\mathrm{tr}(A_1^2A_3)=\mathrm{tr}(A_3^2A_2)=\mathrm{tr}(A_2^2A_1)=0$. The statement follows from Lemma 11.

Let $\mathrm{rank}(A_i)=2$ for every $i=\overline{1,3}$. Conjugating, we can assume that $A_1=J_2$ (Lemma 5). Let $A_2=T_2J_2T_2^{-1}$, $A_3=T_3J_2T_3^{-1}$, where $T_2,T_3\in GL_3(K)$. Lemma 6 implies that there are three possibilities for A_2 and three possibilities for A_3. Instead of letters t_i ($i=\overline{1,5}$) we will use letters b_i for matrix A_2 and letters c_i for matrix A_3 ($i=\overline{1,5}$). Denote by (j,k) the case when the j^{th} possibility is valid for A_2 and the k^{th} possibility is valid for A_3.

Case $(1,1)$. Here A_2, A_3 are strictly upper triangular matrices and the required is obvious.

Case $(1,2)$. The equation (4) implies $1=0$. It is a contradiction, thus the case is impossible.

Case $(1,3)$. The equation (4) implies $1=0$; a contradiction.

Case $(2,1)$. The equation (4) implies $b_3=0$. Thus $\det(T_2)=0$; a contradiction.

Case $(2,2)$. The equation (4) implies $b_2=b_4b_5-\beta_2b_3(-c_2+c_4c_5)$. Thus (6) shows that $c_3=0$ or $c_4=b_4$. In the first case $\det(T_3)=0$, and in the second case the equation (3) implies $1=0$; a contradiction.

Case $(2,3)$. The equation (4) gives $b_2 = b_4b_5 + \beta_2b_3c_2$, thus the equation (6) implies $b_1 = b_4^2 + c_1 - b_4c_3$. The equation (3) implies $c_4 = \beta_1b_3c_2$. It follows from (2) that $\alpha_1 + \beta_1 + \alpha_2\beta_2 = 0$; a contradiction.

Case $(3,1)$. The equation (4) implies $\beta_2 = 0$; a contradiction.

Case $(3,2)$. The equation (4) implies $b_4 = \beta_2(c_2 - c_4c_5)$. Then the equation (6) gives two possibilities:

a) $c_3 = 0$. Thus the equation (3) gives a contradiction.

b) $b_1 = c_1 + (b_3 - c_4)c_4$. Thus the equation (3) gives $c_2 = (\beta_1/\beta_2)b_2c_3 + c_4c_5$. The equation (2) implies $\alpha_1 + \beta_1 + \alpha_2\beta_2 = 0$; a contradiction.

Case $(3,3)$. The equation (6) implies $c_3 = b_3$. Then the equation (3) gives a contradiction. $\triangle$

By the same way it was proved that if we slightly change h.s.o.p.-s of $R_{3,2}$, $R_{4,2}$ for $p = 0$ from [9], then we get h.s.o.p.-s for arbitrary p:

Proposition 2 *1. The set* $\{\sigma_k(X_i)$ *(*$i = 1,2$, $k = \overline{1,3}$*)*, $\mathrm{tr}(X_1X_2)$, $\mathrm{tr}(X_1^2X_2)$, $\mathrm{tr}(X_1X_2^2)$, $\mathrm{tr}(X_1^2X_2^2)\}$ *is a h.s.o.p. of* $R_{3,2}$.

2. The set $\{\sigma_k(X_i)$ *(*$i = 1,2$, $k = \overline{1,4}$*)*, $\mathrm{tr}(X_1X_2)$, $\mathrm{tr}(X_1^2X_2)$, $\mathrm{tr}(X_1X_2^2)$, $\mathrm{tr}(X_1^3X_2)$, $\mathrm{tr}(X_1X_2^3)$, $\mathrm{tr}(X_1^2X_2^2)$, $\sigma_2(X_1X_2)$, $\sigma_2(X_1X_2^2)$, $\sigma_2(X_1^2X_2)\}$ *is a h.s.o.p. of* $R_{4,2}$.

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